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Comments on Entanglement Entropy in the dS/CFT Correspondence

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Abstract

We consider the entanglement entropy in the dS/CFT correspondence. In Einstein gravity on de Sitter spacetime we propose the holographic entanglement entropy as the analytic continuation of the extremal surface in Euclidean anti-de Sitter spacetime. Even though dual conformal field theories for Einstein gravity on de Sitter spacetime are not known yet, we analyze the free $Sp(N)$ model dual to Vasiliev's higher spin gauge theory as a toy model. In this model we confirm the behaviour similar to our holographic result from Einstein gravity.

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1 Introduction

The AdS/CFT correspondence provides a remarkable connection between gravitational theories in anti-de Sitter spacetime (AdS) and nongravitational theories [1–3]. This enables us to analyze quantum gravitational theories by using nongravitational theories.

A useful quantity to analyze gravitational theories is the holographic entanglement entropy proposed in [4, 5][2]. The holographic entanglement entropy contains information on gravitational theories [7, 8]. For instance, Einstein's equation can be reproduced from the holographic entanglement entropy [9, 10].

It is natural to apply the AdS/CFT correspondence to our Universe. However, since it is known that our Universe is approximately de Sitter spacetime (dS), not AdS, we cannot use the AdS/CFT correspondence to analyze our Universe.

The dS/CFT correspondence has been proposed in [11–13]. These proposals were abstract, and concrete examples did not exist. Recently, Anninos, Hartman and Strominger have proposed a concrete example of the dS/CFT correspondence based on Giombi-Klebanov-Polyakov-Yin duality (the duality between Vasiliev's four-dimensional higher-spin gauge theory on Euclidean AdS (EAdS) and the three-dimensional $O(N)$ vector model) [14] (see also [15] for a review). The authors showed that EAdS and the $O(N)$ vector model are related to dS and the $Sp(N)$ vector model via analytic continuation, respectively. It follows that Vasiliev's higher-spin gauge theory on dS is the holographic dual of the Euclidean $Sp(N)$ vector model which lives in $\mathcal{I}^+$ in dS. We are now in a position to analyze the dS/CFT correspondence using the concrete example.

In this paper, we investigate the connection between bulk geometry and holographic entanglement entropy in Einstein gravity on dS. However, the notion of extremal surfaces whose boundaries sit on $\mathcal{I}^+$ is obscure. If the surfaces were space-like, their area would be smaller and smaller as the surfaces approached null. If the surfaces were time-like, their area would be imaginary, and the surfaces would not be closed in general. We discuss this issue based on analytic continuation because the analytic continuation enables us to obtain surfaces in dS, which satisfy the equation of motion obtained from the variation of the area functional.

[2]The covariant generalisation was proposed in [6].

The organization of this paper is as follows. In section 2 we propose the holographic entanglement entropy formula for Einstein gravity on dS. We find extremal surfaces in dS based on a double Wick rotation from EAdS in Poincaré coordinates. We comment on extremal surfaces in more general set of asymptotically dS. In section 3 we calculate the entanglement entropy in the free Euclidean $Sp(N)$ model. We compare this result with the result in section 2 and confirm that our proposal is sensible qualitatively. Section 4 is devoted to a conclusion and discussion.

2 Proposal for Holographic Entanglement Entropy in Einstein Gravity on dS

It is known that a black hole's (BH) entropy is given by an event horizon area divided by four times Newton's constant. The BH entropy formula holds not only in asymptotically flat or AdS but also in asymptotically dS. The holographic entanglement entropy is a generalised quantity of the BH entropy [16], and it is given by an area of a extremal surface divided by four times Newton's constant. It is natural to expect that the Ryu-Takayanagi formula holds even in dS as the BH entropy formula.

Thus our task is to find "extremal surfaces" in Einstein gravity on dS. However, the notion of the extremal surface is obscure as noted in the Introduction. Our proposal is that the extremal surfaces in dS are given by the analytic continuation of extremal surfaces in EAdS. This proposal allows for extremal surfaces which extend in complex-valued coordinate spacetime, and lets us find complex surfaces as extremal surfaces[3]. In the next section, we will check the consistency of our proposal.

The metric of dS in Poincaré coordinates is given by

$$\mathrm{d}s^2 = \ell_{\mathrm{dS}}^2 \frac{-\mathrm{d}\eta^2 + \sum_{i=1}^{d} \mathrm{d}x_i^2}{\eta^2} \tag{2.1}$$

[3] The extremal surfaces in our proposal are space-like. However, it would be possible to regard time-like surfaces as the extremal surfaces. The time-like surfaces might not be appropriate for the holographic entanglement entropy formula because these surfaces are not closed in general. If we consider that we attach a half sphere to a half of dS so that it represents the Hartle-Hawking state, the time-like surfaces would be closed. In this case, the holographic entanglement entropy would become a sum of a pure real part and a pure imaginary part. Since this result largely disagrees with our result in section 3, we will not consider this possibility in this paper.

where η is the conformal time, and $\ell_{\rm dS}$ is a dS radius. By performing a double Wick rotation,

$$\eta \to iz\,, \quad \ell_{\rm dS} \to i\ell_{\rm AdS}\,, \tag{2.2}$$

the metric (2.1) becomes the metric in the Poincaré EAdS,

$$\mathrm{d}s^2 = \ell_{\rm AdS}^2 \frac{\mathrm{d}z^2 + \sum_{i=1}^{d} \mathrm{d}x_i^2}{z^2}\,. \tag{2.3}$$

Here z is a radial direction, and $\ell_{\rm AdS}$ is an AdS radius.

According to the Ryu-Takayanagi formula, the holographic entanglement entropy of a half plane is given by

$$S_A = \frac{V_{d-2}}{4G_{\rm N}} \int_{\varepsilon}^{\infty} \mathrm{d}z \left(\frac{\ell_{\rm AdS}}{z}\right)^{d-1} = \frac{V_{d-2}\ell_{\rm AdS}^{d-1}}{4G_{\rm N}(d-2)} \cdot \frac{1}{\varepsilon^{d-2}} \tag{2.4}$$

where $G_{\rm N}$ is Newton's constant, and ε is a UV cutoff. This is the result in the AdS/CFT correspondence.

Performing the double Wick rotation (2.2) while Newton's constant $G_{\rm N}$ and the UV cutoff ε are held fixed, the above entanglement entropy (2.4) becomes

$$S_A = (-i)^{d-1} \frac{V_{d-2}\ell_{\rm dS}^{d-1}}{4G_{\rm N}(d-2)} \cdot \frac{1}{\varepsilon^{d-2}}\,. \tag{2.5}$$

In the AdS case, the extremal surface is $0 \le z < \infty$ at $x_1 = 0$. After the double Wick rotation, the extremal surface is given by

$$0 \le \eta < i\infty\,. \tag{2.6}$$

The extremal surface in dS is not real-valued but *complex*-valued. The idea of complex surfaces has also appeared in the AdS case [17].

2.1 Extremal surfaces in asymptotically dS

In the previous subsection, we found the extremal surface in Poincaré dS using the double Wick rotation. We comment on extremal surfaces in a more general set of asymptotically dS.

To define extremal surfaces in asymptotically dS, we need to find a double Wick rotation between the asymptotically dS and the corresponding asymptotically EAdS. One Wick rotation is

$$\ell_{\rm dS} \to i\ell_{\rm AdS} \tag{2.7}$$

to make the cosmological constant positive. A second analytic continuation is concerned with a time coordinate in dS.

Our proposal is that the holographic entanglement entropy in the dS/CFT correspondence is defined as

$$S_A := \frac{\mathrm{Area}_{\mathrm{dS}}}{4G_{\mathrm{N}}}\,. \tag{2.8}$$

Here $\mathrm{Area}_{\mathrm{dS}}$ is the area of the "extremal surfaces" in asymptotically dS and is defined as follows. First of all, we find extremal surfaces in asymptotically EAdS. Next, performing the double Wick rotation of the extremal surfaces in asymptotically EAdS, we define "extremal surfaces" $\mathrm{Area}_{\mathrm{dS}}$ in asymptotically dS. As in the previous subsection, the extremal surfaces in dS are *complex*-valued in general although the extremal surfaces in AdS are real-valued. The holographic entanglement entropy (2.8) is uniquely defined by using the extremal surfaces in asymptotically EAdS.

3 Comparison with A Toy CFT Model

Since the conformal field theory (CFT) dual to Einstein gravity on dS is not known yet, we analyze the free $Sp(N)$ model as a toy model. Since the $Sp(N)$ model is the holographic dual of Vasiliev's higher-spin gauge theory on dS, we can quantitatively compare such results only with Vasiliev's higher-spin gauge theory, not with Einstein gravity. However, it is natural to expect that their basic qualitative behaviours do not change between these two theories.

The free $Sp(N)$ model on a Euclidean space with the metric g_{ij} is defined by the action

$$I = \int \mathrm{d}^d x\,\sqrt{g}\,\Omega_{ab} g^{ij}\partial_i\chi^a\partial_j\chi^b\,, \qquad \Omega_{ab} = \begin{pmatrix} 0 & 1_{N/2\times N/2} \\ -1_{N/2\times N/2} & 0 \end{pmatrix}\,, \tag{3.1}$$

where χ^a $(a = 1, \cdots, N)$ are anticommuting scalars, and N is an even integer [18]. By introducing

$$\eta^a = \chi^a + i\chi^{a+\frac{N}{2}}\,, \quad \bar{\eta}^a = -i\chi^a - \chi^{a+\frac{N}{2}} \qquad \left(a = 1, \cdots, \frac{N}{2}\right)\,, \tag{3.2}$$

the action is rewritten as

$$I = \int \mathrm{d}^d x\,\sqrt{g}\,g^{ij}\partial_i\bar{\eta}\,\partial_j\eta\,. \tag{3.3}$$

Let us calculate the "entanglement entropy" in the free $Sp(N)$ model on $\mathbb{R}^d$ ($g_{ij} = \delta_{ij}$). We divide the x_1 slice of $\mathbb{R}^d$ into two regions A and B, and define the entanglement entropy S_A as,

$$S_A := -\mathrm{Tr}_A\,\rho_A \log \rho_A \tag{3.4}$$

Here the reduced density matrix ρ_A is defined as $\rho_A = \mathrm{Tr}_B\,\rho$ by using the total density matrix ρ. For simplicity, we take $x_2 \geq 0$ as the subsystem A. By using the replica trick, the entanglement entropy can be expressed as

$$S_A = -\lim_{n\to 1} \frac{\partial}{\partial(1/n)} \left(\log Z_{\mathbb{R}^2/\mathbb{Z}_n\times\mathbb{R}^{d-2}} - \frac{1}{n}\log Z_{\mathbb{R}^d} \right) , \tag{3.5}$$

where $Z_{\mathbb{R}^2/\mathbb{Z}_n\times\mathbb{R}^{d-2}}$ and $Z_{\mathbb{R}^d}$ are partition functions on $\mathbb{R}^2/\mathbb{Z}_n \times \mathbb{R}^{d-2}$ and $\mathbb{R}^d$, respectively. The logarithm of the partition function is evaluated as

$$\log Z_{\mathbb{R}^d} = \log \int \mathcal{D}\bar{\eta}\,\mathcal{D}\eta\,\mathrm{e}^{-I} = NV_d \log \int \frac{\mathrm{d}^d k}{(2\pi)^d} \log k^2 , \tag{3.6}$$

where V_d is the volume of $\mathbb{R}^d$. Note that this result is minus that of standard field theories. It comes from the statics of the fields. Since $\log Z_{\mathbb{R}^2/\mathbb{Z}_n\times\mathbb{R}^{d-2}}$ is also minus that of the standard field theories, the entanglement entropy is given by

$$S_A = -\frac{NV_{d-2}}{6(d-2)(4\pi)^{\frac{d-2}{2}}} \cdot \frac{1}{\varepsilon^{d-2}} , \tag{3.7}$$

where V_{d-2} is a $(d-2)$-dimensional infinite volume, and ε is a UV cutoff. The entanglement entropy (3.7) is minus that of standard field theories. We can also obtain similar results for arbitrary subsystems on $\mathbb{R}^d$ or other curved spaces.

The result (3.7) holds for any dimensions. However note that it is known that the duality between Vasiliev's higher-spin gauge theory and the $Sp(N)$ model holds only when $d = 3$.

4 Conclusion and discussion

In this paper, we have discussed the entanglement entropy in the dS/CFT correspondence. We have proposed the holographic entanglement formula for Einstein gravity on dS and asymptotically dS via the double Wick rotation. In our proposal we found extremal surfaces which extend in complex-valued coordinate spacetime. The holographic entanglement entropy behaves as $S_A \propto (-i)^{d-1}$ in dS_{d+1} (2.5).

To check that our proposal works, we have calculated the entanglement entropy in the free $Sp(N)$ model and compared it with our proposal. We have found that the entanglement entropy is given by minus that of standard field theories (3.7). This result may suggest that the $\mathrm{dS}_{d+1}/\mathrm{CFT}_d$ correspondence makes senses only when $d \in 4\mathbb{Z} - 1$. Note that the most interesting case, the $\mathrm{dS}_4/\mathrm{CFT}_3$ correspondence, is included, while the most simple case, the $\mathrm{dS}_3/\mathrm{CFT}_2$ correspondence, is excluded. This is consistent with results in subsection 5.2 in [13].

Our proposal has been checked only in the simple case, where the subsystem is the half plane. We need to check our proposal in more nontrivial setups with circular or some other shaped entanglement surfaces for example. However, it is expected that our proposal holds in any entanglement surfaces because the factor $(-i)^{d-1}$ appears in Einstein gravity via the double Wick rotation (2.5), and the minus sign appears in the $Sp(N)$ model (3.7). It is interesting to apply our proposal to Schwarzschild dS. Naively, it is expected that the holographic entanglement entropy is a sum of that of pure dS and the Schwarzschild BH entropy.

Finally, we comment on the negativity of the entanglement entropy (3.7) in the free $Sp(N)$ model. In standard field theories, entanglement entropies are positive definite. In contrast, our result (3.7) is negative definite. The negativity comes from the fact that the scalars of the $Sp(N)$ model are anticommuting, and implies that the inner products of the Hilbert space are not positive definite. This negativity might be a key ingredient of the dS/CFT correspondence.

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Note added:

While this work was in progress, the article [19] appeared in the arXiv. Our results in section 2 are overlapped with [19].

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